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SYNTHESIS OF FRACTIONAL ORDER FUZZY OBSERVER FOR A FRACTIONAL ORDER TAKAGI-SUGENO MODEL WITH UNMEASURABLE PREMISE VARIABLES

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In this article, we present a novel approach: a fractional-order fuzzy observer, an extension of the Thau-Luenberger observer, specifically designed for nonlinear systems characterized by commensurate non-integer order Takagi-Sugeno models. This work makes significant contributions in two key areas. Firstly, both the activation functions of the model and the Lipschitz fractional-order fuzzy observer are dependent on unmeasurable variables, particularly the system's state. Secondly, our proposed fuzzy observer explicitly incorporates the system's initial conditions. The stability conditions of the fractional-order fuzzy observer are expressed through Bilinear Matrix Inequalities, which are then converted into linear matrix inequalities (LMIs). Subsequently, numerical simulations are conducted to demonstrate the efficacy of our proposed estimator.

1. INTRODUCTION

State estimation methodologies for systems characterized by fractional-order linear models have attained a notable level of maturity [1–4]. Nevertheless, the intrinsic linearity inherent in the models delineating the monitored system introduces a pivotal assumption, thereby constraining the applicability and significance of the acquired findings.

The direct extrapolation of methodologies devised for fractional-order linear models to their nonlinear counterparts presents a nuanced challenge [5,6]. Nonetheless, auspicious outcomes have been realized through the adoption of a modeling paradigm predicated on a series of elementary structured models. Each model delineates the system's dynamics within a designated "operational zone." In this context, the Takagi-Sugeno model, renowned for its comprehensive portrayal of system behavior via the interpolation of local linear models, has yielded substantial achievements [7,8].

Within the domain of fractional-order linear models, the regulation and identification of anomalies can be conducted utilizing methodologies grounded in state observers [9–13].

In systems delineated by Takagi-Sugeno models, the application of observer bank schemes for fault diagnosis encounters challenges due to the introduced couplings within the structure. Consequently, limited research endeavors have addressed the development of observers grounded in fractional-order fuzzy Takagi-Sugeno models [14–16].

The state estimation of a continuous nonlinear system can be attained through diverse categories of estimators or observers [16–18]. Herein, we provide a synthesis that rationalizes the selection of the estimator advocated in this study.

The Extended Kalman Filter (EKF) serves as a valuable instrument for handling noisy systems characterized by statistical properties. However, despite its utility, the EKF is subject to several limitations. These encompass the lack of definitive evidence regarding its convergence, the absence of assured reconstruction velocity, its localized applicability primarily around nominal trajectories, and the computational overhead associated with frequent online updates of state estimates and covariance matrices [18,19].

Observers rooted in Lyapunov stability principles offer the advantage of facile implementation, contingent upon the availability of an apt gain matrix to ensure the stability of estimation error dynamics. However, these observers encounter difficulties with the intricate structure of nonlinear systems. The selection of an appropriate gain matrix, meeting the stability criterion for estimation error dynamics, frequently necessitates a trial-and-error approach. This endeavor can pose challenges and, in certain instances, may be unachievable, particularly for high-order systems [19,20].

Observers formulated in canonical form offer a significant advantage: the simplification of observer synthesis into a linear framework after the transformation process. However, these methods encounter challenges in delineating systems possessing the requisite canonical observability form. Moreover, identifying a transformation that fulfills these criteria can be elusive at times, owing to the restricted class of systems amenable to such transformations [19,21].

High-gain observers provide the advantage of achieving rapid convergence of state estimation errors towards zero by modulating the convergence speed through the parameter θ. Nevertheless, excessively large gains may amplify sensitivity to measurement noise, thereby posing a drawback. Additionally, while a coordinate transformation theoretically facilitates the attainment of a triangular structure, practical hurdles persist in its execution, thereby complicating the process [19,22].

Adaptive observers provide concurrent estimation of states and parameters, thereby augmenting robustness against parametric variations in contrast to state observers employing fixed parameter values. However, analogous to canonical observers, they exhibit analogous limitations, as previously delineated [19,23].

Observers grounded in extended linearization endeavor to identify the function h to uphold the invariance of error dynamics' eigenvalues concerning ε. Nonetheless, this approach entails significant drawbacks. Primarily, analytical computation proves burdensome, notably due to integrations, exacerbated further in scenarios with multiple inputs. Secondly, the pursuit of equilibrium points presents a considerable challenge [19,24].

Sliding mode observers are devised to mitigate modeling uncertainties by integrating a complementary term contingent on the output error.

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Several methodologies have been proposed to alleviate chattering, albeit at the cost of permitting non-zero estimation errors. However, adopting this strategy entails a structural assumption concerning the nonlinear function, imposing a significant constraint in numerous scenarios [19,25].

Observers founded on estimation error optimization address an optimization dilemma, usually employing gradient methods, which, unfortunately, require substantial computational resources, primarily due to the need for solving it at each sampling instance. Nonetheless, this approach presents a noteworthy advantage in its applicability to a diverse range of nonlinear systems [19,26].

The state-of-the-art review of existing observer synthesis methodologies has offered insights into the merits and demerits of each approach.

In summary, certain methodologies are intricately linked to specific attributes of nonlinear systems, exemplified by high-gain observers and those reliant on Lyapunov techniques. Conversely, other approaches overlook uncertainties and modeling inaccuracies, as evidenced by the extended Kalman filter, observers based on extended linearization, high-gain observers, or those formulated in canonical form. Moreover, methodologies such as optimization-based, sliding mode, or adaptive observers necessitate substantial computational resources.

The study proposes Fractional Order Fuzzy Observers customized for Takagi-Sugeno (T-S) models, amalgamating principles derived from fractional calculus, fuzzy logic, and multiple-model methodologies. By incorporating fractional order dynamics, these observers effectively capture nuanced system behaviors, encompassing memory effects and long-range dependencies. Harnessing a multiple-model framework, they inherently adapt to fluctuating operating conditions and uncertainties within the model structure, thus augmenting the robustness and adaptability of state estimation in T-S fuzzy systems [27].

Broadly, the formulation of a non-integer order observer for a system delineated by a non-integer order Takagi-Sugeno fuzzy model entails the design of fractional-order local observers, followed by their interpolation according to predefined weight functions. This design methodology extends the scope of analysis and synthesis techniques, originally formulated for linear non-inter order systems, to encompass the nonlinear non-integer order domain [27].

Notably, in [5,28], observers devised for linear systems have been adjusted for application to Takagi-Sugeno models. However, it's crucial to note that these investigations primarily focus on Takagi-Sugeno fuzzy models utilizing activation functions contingent on measurable decision variables (system input or output), a characteristic also shared by the fractional-order fuzzy observer.

In the literature, sparse efforts have been made regarding the development of fractional-order fuzzy state observers with activation functions reliant on non-measurable variables (such as system state) [27].

Our contribution in this article resides in the state estimation of a nonlinear system delineated by a fractional-order Takagi-Sugeno fuzzy model, incorporating activation functions dependent on unmeasurable decision variables (system states).

The stability conditions of the non-integer order fuzzy observer are determined using a quadratic Lyapunov function, which is represented by a set of linear matrix Inequalities (LMIs).

2. FRACTIONAL ORDER SYSTEMS

The state-space model presented below [29] represents the non-integer order linear system:

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} x(t) = A_i x_i(t) + B_i u(t), \\ y(t) = C_i x_i(t), \end{cases} \tag{1}$$

where $x_i(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, and $y(t) \in R^p$ is the output vectors of the system and $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$, $C_i \in R^{p\times n}$, and $\alpha \in [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ are the fractional orders.

${}_a\mathrm{D}_t^{\alpha} f(t)$ denotes the fractional differ-integral operators, where a and t are the bounds of the operation and $\alpha \in R$ is a generalization of the standard integration and differentiation to an arbitrary order, which can be rational, irrational, or even complex. The basic continuous differ-integral operator is given as follows [27]:

$$ {}_a\mathrm{D}_t^{\alpha} = \begin{cases} \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} & \text{for } \alpha > 0, \\ 1 & \text{for } \alpha = 0, \\ \int_{\alpha}^{t} (\mathrm{d}\tau)^{\alpha} & \text{for } \alpha < 0. \end{cases} \tag{2}$$

In the literature, we can find different definitions related to fractional order systems. The most commonly used definitions of fractional order derivatives are:

The Riemann-Liouville (RL) definition [27]:

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} \mathrm{d}\tau. \tag{3}$$

The Grunwald-Letnikov (GL) definition [24]:

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{(t-a)/h} (-1)^k \binom{\alpha}{k} f(t-kh). \tag{4}$$

The fractional differo-integral operator for the function $f(t)$ following Caputo definition is adopted in this paper because it allows utilization of the initial values of classical integer-order derivatives with clear physical interpretation as follows [30,31]:

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{m}(\tau)}{(t-\tau)^{1-(m-\alpha)}} \mathrm{d}\tau. \tag{5}$$

In these expressions, $m-1 < \alpha < m$, and $\Gamma(.)$ is the Euler's gamma function [32]:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{(x-1)} \mathrm{d}t, \ \ x > 0. \tag{6}$$

In this work, we will examine the commensurate non-integer order nonlinear system, which is given by [33]:

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} x_i(t) = f_i(x_1(t), x_2(t), \ldots, x_n(t), t), \\ x_i(0) = c_i, \quad i = 1,2,\ldots,n, \end{cases} \tag{7}$$

where c_i represents the initial conditions. The vector form of eq (7) is:

$$ {}_a\mathrm{D}_t^{\alpha} x(t) = f(x), \tag{8}$$

where, $f(x)$ is the nonlinear function that determines the behavior of $x(t)$, $x(t) \in R^n$, and $\alpha \in R$ represents a broad extension surpassing conventional integration and differentiation, accommodating orders that are not limited to rational or irrational numbers but can also include complex numbers.

The equilibrium points of system (8) can be ascertained by resolving the subsequent equation:

$$f(t) = 0. \tag{9}$$

We posit that a designated equilibrium point of the system is established as:

$$E^{*} = (x_1^{*}, x_2^{*}, \dots, x_n^{*}). \tag{10}$$

where E^{*} defines the equilibrium point of the system as a vector wherein each component signifies the equilibrium value of the respective variable within the system, n is the dimensionality of the system, $x_1^{*}, x_2^{*}, \dots, x_n^{*}$ are the individual components of the equilibrium point vector. Each component x_i^{*} represents the value of the i^{th} variable of the system at equilibrium.

For ease of writing, in the subsequent equations, we use f instead of $f(t)$ right after this relation.

3. FRACTIONAL ORDER TAKAGI-SUGENO MODEL APPROACH

The ability of Takagi-Sugeno fuzzy models to represent or approximate the dynamic behavior of a real system has been widely recognized. Indeed, on one hand, they offer the possibility to describe highly complex nonlinear behaviors with a simple structure inspired by linear models.

On the other hand, their particular structure allows for the extension of certain results obtained within the framework of linear systems [34].

T-S models are the most studied in the literature; they are described by a set of sub-models sharing a unique state vector.

Two categories can be delineated based on the nature of the variables incorporated in the weight functions. These variables, denoted as decision variables or premise variables, can either be known (system input or output, *etc.*) or unknown (system state, *etc.*).

The category of Takagi-Sugeno (T-S) fuzzy models with measurable decision variables (MDV) has undergone extensive advancements across diverse domains, notably in control, stabilization, and state estimation [35], and diagnostics. However, the second category is very little explored, especially in the field of fuzzy observer design and their use for diagnostics. Consider a non-integer order Takagi-Sugeno fuzzy model where the activation functions h_i depend on the state of the system:

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} x = \sum_{i=1}^{N} h_i(x)(A_i x + B_i u), \\ \quad y = Cx, \end{cases} \tag{11}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the input vector, $y \in R^p$ is the output vector. $A_i \in R^{n\times n}$ is a state matrix, $B_i \in R^{n\times m}$ is an input influence matrix and $C_i \in R^{p\times n}$ is the output or observation matrix. N is the number of sub-linear models. $\alpha \in R^n$ is the fractional order satisfying. Finally, the activation functions h_i depend on a non-measurable variable (system state), and verify:

$$\begin{cases} \sum_{i=1}^{N} h_i(x) = 1, \\ 0 \le h_i(x) \le 1 \quad \forall\, i \in \{1,2,\dots,n\}. \end{cases} \tag{12}$$

4. FRACTIONAL ORDER FUZZY OBSERVER SYNTHESIS

The subsequent form of fuzzy observer is proposed:

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} \hat{x} = \sum_{i=1}^{N} h_i(\hat{x})\big(A_i \hat{x} + B_i u + L_i(y - \hat{y})\big), \\ \hat{y} = C\hat{x}, \end{cases} \tag{13}$$

where ${}_a\mathrm{D}_t^{\alpha}\hat{x}$ is the fractional order derivative of the estimated state vector $\hat{x}$ with respect to time. $\hat{y}$ is the estimated of the measured output vector.

L_i is the gain matrix associated with each rule i.

This work introduces a novel method enabling the state estimation of the system (11).

For this, we pose $A_i = A_0 + \overline{A}_i$, where A_0 is given as follows:

$$A_0 = \frac{1}{N}\sum_{i=1}^{N} A_i. \tag{14}$$

By replacing the structure of A_i in the model eq. (11), we obtain:

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} x = A_0 x + \sum_{i=1}^{N} h_i(x)\,\big(\overline{A}_i x + B_i u\big), \\ y = Cx. \end{cases} \tag{15}$$

From this model, we propose the following fractional order fuzzy observer by replacing A_i in (13):

$$\begin{cases} {}_a\mathrm{D}_t^{\alpha} \hat{x} = A_0\hat{x} + \sum_{i=1}^{N} h_i(\hat{x})\big(\overline{A}_i \hat{x} + B_i u + L_i(y - \hat{y})\big), \\ \qquad \hat{y} = C\hat{x}. \end{cases} \tag{16}$$

The advantage of this representation compared to (11) is to obtain fewer conservative conditions of existence.

The state estimation error and the output estimation error can be delineated as follows:

$$\begin{cases} e = x - \hat{x}, \\ \tilde{y} = y - \hat{y}. \end{cases} \tag{17}$$

The dynamics of the state estimation error are given by:

$$\begin{aligned} {}_a\mathrm{D}_t^{\alpha} e = \textstyle\sum_{i=1}^{N}\big(h_i(x)(A_i x + B_i u) \\ - h_i(\hat{x})(A_i\hat{x} + B_i u + L_i C e)\big). \end{aligned} \tag{18}$$

$${}_a\mathrm{D}_t^{\alpha} e = \sum_{i=1}^{N} h_i(\hat{x})(A_i - L_i C)e + \varepsilon(x,\hat{x},u), \tag{19}$$

where:

$$\varepsilon(x,\hat{x},u) = \sum_{i=1}^{N}\big(h_i(x) - h_i(\hat{x})\big)(A_i x + B_i u), \tag{18}$$

where the gains L_i are determined by solving a set of LMI involving h the Lipschitz constant of term (20). Unfortunately, for a high value of the Lipschitz constant h, the LMI system to be solved may not have a solution.

The dynamics of the estimation error is obtained with eqs. (15) and (16):

$${}_a\mathrm{D}_t^{\alpha} e = \sum_{i=1}^{N} h_i(\hat{x}) N_i e + \sum_{i=1}^{N} \overline{A}_i\, v_i + \sum_{i=1}^{N} B_i \varepsilon_i\,, \tag{21}$$

where

$$\begin{cases} N_i = A_0 - L_i C, \\ v_i = h_i(x)x - h_i(\hat{x})\hat{x}, \\ \varepsilon_i = \big(h_i(x) - h_i(\hat{x})\big)u. \end{cases} \tag{22}$$

***Theorem* 1.** *The state estimation error between the non-integer order Takagi-Sugeno system (15) and the non-integer order fuzzy observer (16) asymptotically tends to zero if there exist two positive definite symmetric matrices P and Q, matrices G_i, and positive scalars λ_1, λ_2, and η such that the following matrix inequalities are satisfied*:

$$A_0^T P + \mathrm{P}A_0 - G_i^T P - \mathrm{P}G_i < -Q. \tag{23}$$

$$\begin{bmatrix} -Q + \lambda_1 T_i^2 I & P\overline{A}_i & PB_i & F_i\eta I \\ \overline{A}_i^T P & -\lambda_1 I & 0 & 0 \\ B_i^T P & 0 & -\lambda_2 I & 0 \\ F_i\eta I & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0. \tag{24}$$

$$\eta - \beta_1\lambda_2 > 0. \tag{25}$$

Proof. For the study of observer convergence, let us make the following assumptions:

Assumption 1. The activation functions are Lipschitz:

$$|h_i(x) - h_i(\hat{x})| \le F_i|x - \hat{x}|, \tag{26}$$

$$|h_i(x)x - h_i(\hat{x})\hat{x}| \le T_i|x - \hat{x}|, \tag{27}$$

where F_i and T_i, positive scalars, are the Lipschitz constants.

Assumption 2. The input u of the system is bounded:

$$\|u\| \leq \beta_1, \quad \beta_1 > 0. \tag{28}$$

To illustrate the stability of the state estimation error, we investigate it using the Lyapunov quadratic function $V(e) = e^T Pe, P > 0$, , whose derivative with respect to time is:

$$ {}_aD_t^\alpha V(e) \leq {}_aD_t^\alpha e^{\mathrm{T}} Pe + e^{\mathrm{T}} P\, {}_aD_t^\alpha e. \tag{29}$$

Then, by using (21):

$$ {}_aD_t^\alpha V \leq \sum_{i=1}^N \Big(v_i^{\mathrm{T}} \overline{A}_i^{\mathrm{T}} Pe + e^{\mathrm{T}} P\overline{A}_i v_i + \varepsilon_i^{\mathrm{T}} B_i^{\mathrm{T}} Pe + +e^{\mathrm{T}} PB_i\varepsilon_i + h_i(\hat{x})\big(e^{\mathrm{T}} N_i^{\mathrm{T}} Pe + e^{\mathrm{T}} PN_i e\big)\Big). \tag{30}$$

Considering (22) and assumptions 1 and 2, we then have:

$$\begin{cases} |v_i| \leq T_i|e|, \\ |\varepsilon_i| \leq F_i\beta_1|e|. \end{cases} \tag{31}$$

***Lemma* 1.** *For all X and Y matrices of appropriate dimensions, λ being a positive constant, the following property holds*:

$$X^{\mathrm{T}} Y + Y^{\mathrm{T}} X \leq \lambda X^{\mathrm{T}} X + \lambda^{-1} Y^{\mathrm{T}} Y, \ \lambda > 0. \tag{32}$$

By applying this lemma as well as (31), we have the following increases:

$$\begin{aligned} v_i^{\mathrm{T}} \overline{A}_i^{\mathrm{T}} Pe + e^{\mathrm{T}} P\overline{A}_i v_i &\leq \lambda v_i^{\mathrm{T}} v_i + \lambda^{-1} e^{\mathrm{T}} P\overline{A}_i\overline{A}_i^{\mathrm{T}} Pe, \\ &\leq \lambda_1 T_i^2 e^{\mathrm{T}} e + \lambda_1^{-1} e^{\mathrm{T}} P\overline{A}_i\overline{A}_i^{\mathrm{T}} Pe, \end{aligned} \tag{33}$$

$$\begin{aligned} \varepsilon_i^{\mathrm{T}} B_i^{\mathrm{T}} Pe + e^{\mathrm{T}} PB_i\varepsilon_i &\leq \lambda_2\varepsilon_i^{\mathrm{T}}\varepsilon_i + \lambda_2^{-1} e^{\mathrm{T}} PB_iB_i^{\mathrm{T}} Pe, \\ &\leq \lambda_2 F_i^2\beta_1^2 e^{\mathrm{T}} e + \lambda_2^{-1} e^{\mathrm{T}} PB_iB_i^{\mathrm{T}} Pe. \end{aligned} \tag{34}$$

The derivative of the Lyapunov function (30) can then be augmented as follows:

$$ {}_aD_t^\alpha V(e) \leq \sum_{i=1}^N e^{\mathrm{T}}\big(h_i(\hat{x})\big)\big(N_i^{\mathrm{T}} P + PN_i\big) +\big(\lambda_1 T_i^2 + \lambda_2 F_i^2\beta_1^2\big)I + \lambda_1^{-1} P\overline{A}_i\overline{A}_i^{\mathrm{T}} P + \lambda_2^{-1} PB_iB_i^{\mathrm{T}} P\Big) e. \tag{35}$$

The negativity of the derivative of the considered Lyapunov function is guaranteed if for $i = 1,2,\dots,n$:

$$h_i(\hat{x})(N_i^T P + PN_i) + \big(\lambda_1 T_i^2 + \lambda_2 F_i^2\beta_1^2\big)I +\lambda_1^{-1} P\overline{A}_i\overline{A}_i^{\mathrm{T}} P + \lambda_2^{-1} PB_iB_i^{\mathrm{T}} P\Big) < 0, \tag{36}$$

which leads to the following conditions:

$$(A_0 - L_iC)^{\mathrm{T}} P + P(A_0 - L_iC) < -Q, \tag{37}$$

$$-Q + \big(\lambda_1 T_i^2 + \lambda_2 F_i^2\beta_1^2\big)I + \lambda_1^{-1} P\overline{A}_i\overline{A}_i^{\mathrm{T}} P +\lambda_2^{-1} PB_iB_i^{\mathrm{T}} P < 0. \tag{38}$$

By changing the variables $G_i = PL_i$, and by applying Schur's complement, we obtain the linear matrix inequalities:

$$A_0^T P + PA_0 - C^T G_i^T - G_iC < -Q, \tag{39}$$

$$\begin{bmatrix} -Q + \big(\lambda_1 T_i^2 + \lambda_2 F_i^2\beta_1^2\big)I & P\overline{A}_i & PB_i \\ \overline{A}_i^{\mathrm{T}} P & -\lambda_1 I & 0 \\ B_i^{\mathrm{T}} P & 0 & -\lambda_2 I \end{bmatrix} < 0. \tag{40}$$

$$\lambda_1 > 0, \quad \lambda_2 > 0$$

Rather than imposing a priori the bound on the input, we can add a degree of freedom by considering this bound as a variable to be determined that we will call ρ. Using Schur's complement, the inequality (40) is written:

$$\begin{bmatrix} -Q + \lambda_1 T_i^2 I & P\overline{A}_i & PB_i & F_i\lambda_2\rho I \\ \overline{A}_i^{\mathrm{T}} P & -\lambda_1 I & 0 & 0 \\ B_i^{\mathrm{T}} P & 0 & -\lambda_2 I & 0 \\ F_i\lambda_2\rho I & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0. \tag{41}$$

$$\lambda_1 > 0, \quad \lambda_2 > 0$$

This inequality is no longer linear in the unknowns (presence of product $\lambda_2\rho$). To write it in LMI form, we put: $\eta = \lambda_2\rho$.

$$A_0^{\mathrm{T}} P + PA_0 - C^{\mathrm{T}} G_i^{\mathrm{T}} - G_iC < -Q. \tag{42}$$

$$\begin{bmatrix} -Q + \lambda_1 T_i^2 I & P\overline{A}_i & PB_i & F_i\eta I \\ \overline{A}_i^{\mathrm{T}} P & -\lambda_1 I & 0 & 0 \\ B_i^{\mathrm{T}} P & 0 & -\lambda_2 I & 0 \\ F_i\eta I & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0. \tag{43}$$

Knowing η and λ_2, we can deduce the value ρ:

$$\rho = \frac{\eta}{\lambda_2}. \tag{44}$$

To ensure that the values of η and λ_2 satisfy hypothesis 2 (ρ must be greater than or equal to β_1), we propose to add a constraint on η and λ_2 to guarantee $\frac{\eta}{\lambda_2} \geq \beta_1$:

$$\eta - \beta_1\lambda_2 \geq 0. \tag{45}$$

Using constraint (45) with conditions (42) and (43), the value found ρ is greater than or equal to the bound of input β_1.

So far, the conditions found only guarantee asymptotic convergence. To ensure certain temporal properties, we propose to ensure the placement of the poles of the matrices $(A_0 - L_iC)$ in specific regions.

5. EXAMPLE

Consider the non-integer Takagi-Sugeno fuzzy system given by:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -6 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 0.5 & 0.5 & -4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0.25 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The activation functions are selected in the following format:

$$\begin{cases} h_1(x) = \frac{1-\tanh(x_2)}{2}, \\ h_2(x) = 1 - h_1(x) = \frac{1+\tanh(x_2)}{2}. \end{cases} \tag{46}$$

The activation functions are and depend only on the first component of the state. Initial condition of the system $x_0 = (0,0,1)$ and for the observer $\hat{x}_0 = (0,0,0)$.

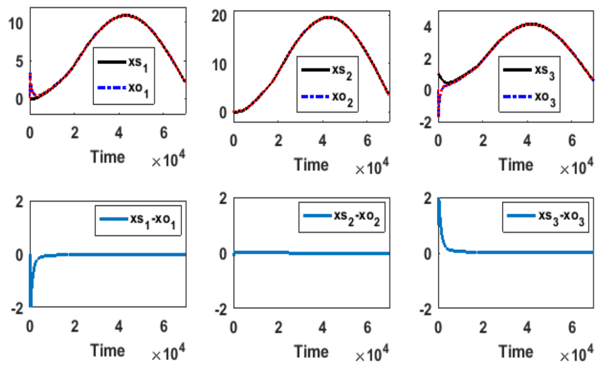

Fig.1 – States estimation and their error of estimation.

Figures 1 and 2 depict the comparison between the state of the non-integer order Takagi-Sugeno fuzzy system and its estimation from the non-integer order fuzzy observer.

Figure 1 shows the three states of the non-integer order system and their respective estimates. Figure 2 illustrates the two outputs of the non-integer order system along with their corresponding estimates.

It is observed that all the plots depicted in Figs. 1 and 2 overlap except in the proximity of the origin. This deviation is attributed to the selection of initial conditions for the non-integer order fuzzy observer.

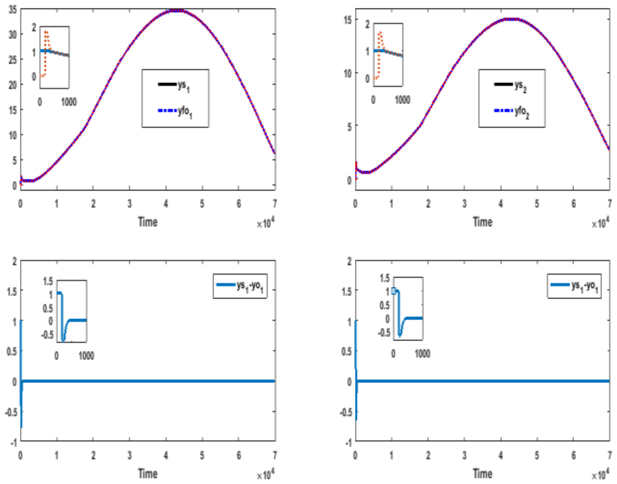

Fig. 2 – Outputs estimation and their error of estimation.

The evaluation of the state estimation error demonstrates that the observer converges towards the system state, with the rapid decrease in error indicating faster convergence. Analyzing the time required for the observer to reach an acceptable proximity to the actual system state substantiates the performance of the developed observer. The observer converges stably, exhibiting minimal oscillations or undesired behavior, which confirms the results of the convergence assessment based on Lyapunov functions. The state estimation curves offer a clear visualization of the developed observer's performance.

6. CONCLUSION

Utilizing a non-integer order Takagi-Sugeno model framework, we propose the design of a non-integer order fuzzy observer through the interpolation principle applied to local fractional-order observers. Additionally, we address scenarios involving non-measurable decision variables. The determination of gains for the fractional-order fuzzy observer is thereby formulated as a simultaneous computation of gains for the local non-integer order observers.

Ensuring the stability of the overall fractional-order fuzzy observer necessitates accounting for coupling constraints among these individual local fractional-order observers. These constraints necessitate resolving a Linear Matrix Inequality (LMI) problem subject to structural constraints. Assuming the availability of appropriate matrices, we demonstrate the feasibility of reconstructing the state and unknown input vectors of the non-integer order Takagi-Sugeno model.

The simulation results demonstrate highly satisfactory state and output estimation performance.

It is important to emphasize that the synthesis of the proposed fractional-order fuzzy observer in this study relies on the validation of two assumptions: (1) and (2). Therefore, as a perspective, we recommend the synthesis of other types of fractional-order fuzzy observers addressing the case of non-satisfaction with one or both assumptions.

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