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THE NUMBER OF POLYNOMIAL SEGMENTS AND THE POLYNOMIAL ORDER OF POLYNOMIAL-BASED FILTERS

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Key words: Decimation, Estimation formula, Farrow structure, Interpolation, Polynomial-based interpolation filters.

Many digital signal processing applications can benefit from polynomial-based interpolation filters based on the Farrow structure or its variations. The number of polynomial segments determining the finite length of the filter impulse response and the order of polynomials in each polynomial segment are the two main design parameters for these filters. These parameters are linked to the complexity of the implementation structure and frequency domain performance. As a result, determining the value of these two parameters based on system requirements is beneficial in order to estimate complexity of the filter, and starting values for a design. This paper offers formulas for estimating the length and polynomial order of polynomial-based filters for a variety of criteria, including stopband attenuation, transition bandwidth, passband deviation, and passband/stopband weighting.

1. INTRODUCTION

In many applications, it is necessary to determine signal samples at arbitrary positions between existing samples of a discrete-time signal, for example, sampling rate conversion for non-integer ratio, timing synchronization, interpolation in image processing [1], etc. In these cases, polynomial-based interpolation filters can be used with piecewise polynomial impulse response [2, 3]. The Farrow structure [4] and its variants [5–7] can be used to efficiently construct these filters in practice. In the literature, several design methods are defined in the time or frequency domain [2, 3, 8–12]. In all these methods, there are two basic parameters, the number of polynomial segments M and polynomial order N, which control the performance of the filter and its complexity. M and N are directly proportional to the number of multipliers in the Farrow structure [2]. As a result, determining the value of these two parameters based on system needs is crucial. In [13], we derived estimation formulas for M and N which are suitable for several cases, but those formulas are obtained experimentally using the trial and error method, thus their accuracy is questionable. The formula presented in [14] is more general, however, it does not cover all possible modifications of the Farrow structure.

In this contribution, we give the estimation for the number of polynomial segments N and polynomial order M. The formulas are developed due to a variety of system criteria, such as the stopband attenuation, the transition bandwidth, the passband deviation, and the passband/stopband weighting. The formulas provided here may be used for different modifications of the Farrow structure, and are a useful starting point for designing polynomial-based filters.

2. POLYNOMIAL-BASED FILTERS AND FARROW STRUCTURE

Polynomial-based filters are characterized by the underlying continuous-time impulse response, $h_a(t)$, whose desirable characteristics, when deriving the modified Farrow structure for interpolation, are [2, 3]:

1) $h_a(t)$ is nonzero for $0 \le t < NT$ and zero elsewhere.
2) In each subinterval $nT \le t < (n+1)T$ for $n = 0,1,\ldots, N-1$, $h_a(t)$ is expressed as a polynomial of degree M.
3) $h_a(t)$ is symmetric about $t = NT/2$, that is $h_a(NT-t) = h_a(t)$.

Based on Characteristic 3), the whole system will have a linear phase, which may be utilized to optimize the overall filter to fulfil the requirements needed in a similar way to linear-phase finite impulse response (FIR) filters [15]. T, the length of the polynomial segments, is not unique in the scenario above. As a result, T may be used to specify a variety of implementation structures, as will be explained later. T can be selected as $T = \beta T_{in}$ or $T = \beta T_{out}$, where β is unity, an integer, or one divided by an integer, as shown in [5, 6]. The choice is made based on whether decimation or interpolation is being considered, as well as the structural requirements for effective implementation.

The implementation structures for polynomial-based interpolation filters have several forms; In the first group are the modified Farrow structure [2, 4] for interpolation and the transposed modified Farrow structure [6] for decimation, where the length of the polynomial segments T is set to the input and output sampling intervals in the interpolation and decimation cases, respectively, *i.e.*, $T = T_{in}$ or $T = T_{out}$. For the second group of structures, T is an integer fraction of the sampling period, *i.e.* $T = \beta T_{in}$ or $T = \beta T_{out}$ with $\beta < 1$, and the structures under consideration are multistage systems consisting of a fixed linear-phase FIR interpolator in cascade with the modified Farrow structure or of a transposed modified Farrow structure in cascade with a fixed linear-phase FIR decimator [5]. Finally, for both interpolation and decimation, the so-called prolonged Farrow structures [5, 6] constitute the third group of implementation forms. The generation of these structures differs from those mentioned above in the fact that T is an integer multiple of either the input or output sampling period, *i.e.* $T = \beta T_{in}$ or $T = \beta T_{out}$ where $\beta > 1$ is an integer. The number of fixed coefficients in all of these structures is determined by the number N of polynomial segments and the order M of the polynomial in each segment.

In this paper, we applied the minimax design method presented in [3]. Let us assume a lowpass signal as an example, whose sampling rate is $F_{in} = 1/T_{in}$, and the sampling rate of the output signal is $F_{out} = RF_{in}$, and in the case of $R > 1$ ($R < 1$) the system realizes interpolation

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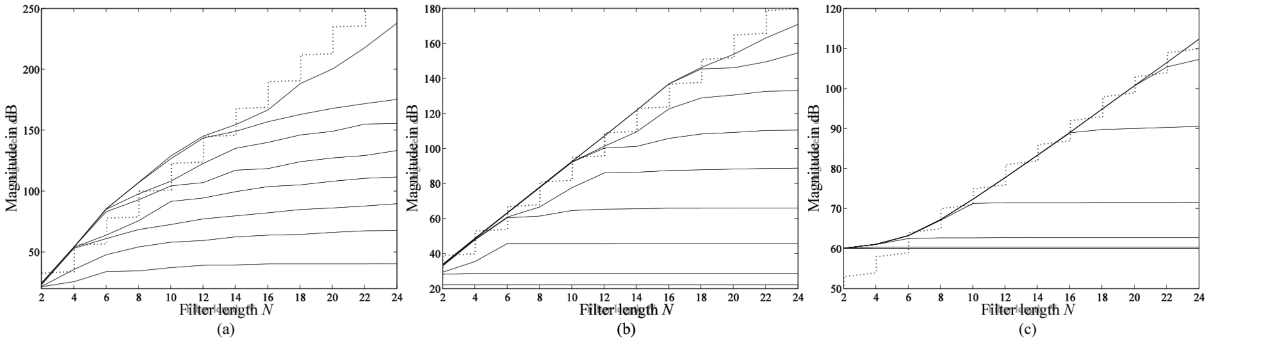

Fig. 1 – *Case C specifications*: The curves are shown for M equals 0 to 7. Dashed line is plot obtained from the estimation formula for N shown in [13]. The stopband edge is at f_s=1-f_p, passband edge and stopband weighting are at: (a) f_p=0.15F and W=0.1; (b) f_p=0.3F and W=10; (c) f_p=0.45F and W=1000.

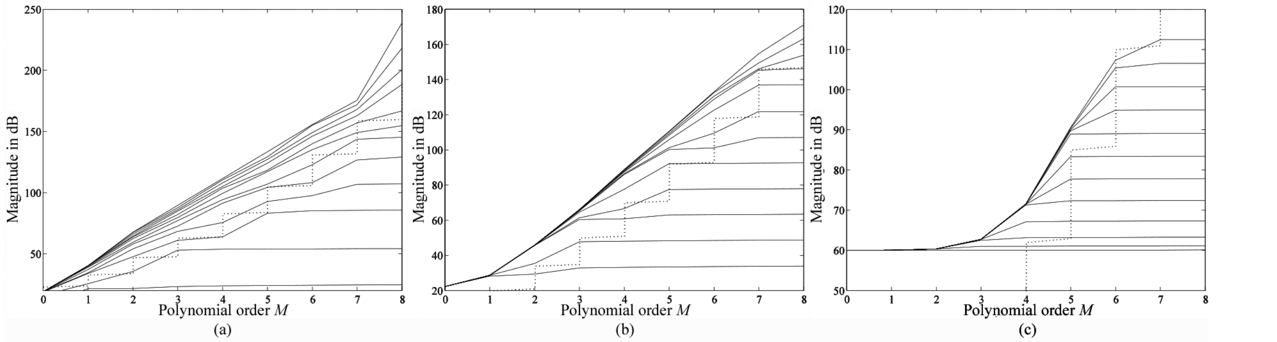

Fig. 2 – *Case C specifications*: The curves are shown for N equals 2 to 24. Dashed line is plot obtained from the estimation formula for M shown in [15]. The stopband edge is at f_s=1–f_p, passband edge and stopband weighting are at (a) f_p= 0.15F and W =0.1; (b) f_p=0.3F and W=10; (c) f_p=0.45F and W=1000.

(decimation). Then the requirements for the zero-phase response $H_a(f)$ are [3]:

$$\begin{aligned} &(1-\delta_p) \le H_a(f) \le (1+\delta_p) \quad |f| \le f_p \\ &H_a(f) \le \delta_s, \; |f| \in \Phi_s, \end{aligned} \tag{1}$$

where

$$\Phi_s = \begin{cases} [F/2, \infty] & Case \;\; A \\ \bigcup_{k=1}^{\infty} [kF - f_p, kF + f_p] & Case \;\; B \\ [F - f_p, \infty] & Case \;\; C. \end{cases} \tag{2}$$

In the above equations, f_p is the passband edge, the stopband region is Φ_S, where k is an integer, δ_s is the minimum stopband attenuation, and δ_p is maximum allowable passband ripple, F stands for F_{out} in a decimation case, and F_{in} in an interpolation case.

In *Minimax Optimization Problem* the length N, and order M, are given, as well as passband and stopband through f_p and Φ_S, weight function $W(f)$. The methods finds the $(M+1)N/2$ unknown coefficients of the Farrow structure to minimize the energy of error between desired and obtained magnitude response.

For the optimization of prolonged and transposed prolonged polynomial-based filters, the design process has been broadened and improved [5]. As a result, for a minimax design, we provide estimate formulas for the number N of polynomial segments and the order M of the polynomial for this case as well.

4. ESTIMATION OF *N* AND *M*

Polynomial-based filters can be represented as FIR filters in practice [16]. Furthermore, we have used a continuous-time Kaiser window to design the polynomial-based interpolation filter in [12]. The estimation formula for N, as found in [17], is insufficiently precise. Both N and M estimation formulas have been proposed in [15]. However, these formulae are produced by a technique of trial and error, and they have some conditional restrictions. In particular, when designing a filter with a small transition band, the estimate equations of [13] cannot be utilized. As a result, we suggest the more universal and precise formulae in this paper.

Similarly to the methodology applied in [14], our starting point is the Kaiser formula for order estimation of FIR filters, and we employed experimentally acquired results to generate estimate formulas. For the following design parameters, the polynomial-based filters are constructed using the minimax optimization technique of [3]. With the step equal to Δf_p = 0.05 normalized to F, the passband edge is altered from Δf_p to $F/2$–Δf_p. The stopband edge is calculated using *Case A* specification of (2), thus f_s = 0.5 normalized to F. $W(f)$, the weighting function used to distinguish design accuracy in the passband and stopband, is also changed from $W(f) = [W_p \; W_s]$ = [1 0.1] to $W(f)$ = [1 1000], where W_p and W_s are passband and stopband weights, respectively.

With step two, N can range from 2 to 24, and M can range from 0 to 7. We utilize 9 distinct values of f_p, five different values of $W(f)$, 12 different values of N, and 8 various values of M, totalling 95128 = 4320 filters. For each set of conditions, we calculated the resulting performance in terms of passband ripple δ_p and stopband ripple δ_s.

Comparing the results in [14] with those in Figs. 1 and 2, one can see that estimation formulas for N and M for *Case A* are valid for *Case C* as well. However, Figs. 3 and 4 show that the *Case B* estimation formulae for N and M should be different from the formulae for *Case A* and *C*.

For a given M, f_p, and $W(f)$, Figs. 1–4 show that there is a certain value of N after which the stopband attenuation δ_s saturates in value, and *vice versa*, for a given N, f_p, and

$W(f)$, there is a certain value of M after which the stopband attenuation δ_s saturates in value. So, these values of N and M are the optimal values that should be used in design for given δ_s, f_p, and $W(f)$. These values and parameters are put in the curve fitting toolbox with the starting formulae for N and M from [14]. The following estimation formulae for *Case A* and *C* are obtained:

$$N = 2\cdot\left\lceil 0.5\cdot\left(\left(\frac{A_s}{28\cdot(f_s-f_p)}\right)+\left(\frac{0.45\cdot(1-\log_{10}(W))}{(f_s-f_p)}\right)-0.5\cdot\log_{10}(W)-0.5\right)\right\rceil, \quad (3)$$

$$M = \left\lceil\sqrt{\frac{A_s+(10\log_{10}(W)-40)\cdot(f_s-f_p)+10}{1.5}+\frac{0.13}{(f_s-f_p)}}\right\rceil-3\cdot \quad (4)$$

where f_p and f_s are the passband and stopband edge, $A_s = -20\log_{10}(\delta_s)$ is the required attenuation in the stopband, $W = \delta_p/\delta_s$ represents weighting between required tolerances in passband and stopband, $\lceil x \rceil$ stands for the smallest integer which is larger or equal to x. If the transition band is small, as in $(f_s - f_p)/F \leq 0.1$, the necessary value of N should be raised by 2.

We also derive the *Case B* estimation formulae:

$$N = 2\left\lceil 0.5\cdot\left(\left(\frac{A_s}{15.5\cdot(f_s-f_p)}\right)-\frac{\log_{10}(W))}{(f_s-f_p)}+\log_{10}(W)-2\right)\right\rceil\cdot \quad (5)$$

$$M = \left\lceil\sqrt{\frac{A_s-4\cdot(\log_{10}(W)^2-8\cdot\log_{10}(W)-15)}{3.5}}+\frac{2.5\cdot(\log_{10}(W)}{3.5}-4(f_s-f_p)\right\rceil+2\cdot \quad (6)$$

When the transition band is very narrow, such as when $(f_s - f_p)/F \leq 0.1$, the estimating formula, like *Case A* and *C*, cannot be utilized.

The above-presented estimation formulae also lead to the estimation formulae for the prolonged Farrow structures and Farrow structures in cascaded multirate filters [16]. Based on (3)–(6), we derive the following *Case A* and *C* in (7) and (8), and *Case B* in (9) and (10) estimation formula for the prolonged Farrow structure:

$$N = 2\left\lceil 0.5\left(\left(\frac{A_s}{28\cdot\beta\cdot(f_s-f_p)}\right)+\left(\frac{0.45\cdot(1-\log_{10}(W))}{\beta\cdot(f_s-f_p)}\right)-0.5\cdot\log_{10}(W)-0.5\right)\right\rceil\cdot \quad (7)$$

$$M = \left\lceil\sqrt{\frac{A_s+(10\log_{10}(W)-40)\cdot\beta\cdot(f_s-f_p)+10}{1.5}+\frac{0.13}{\beta\cdot(f_s-f_p)}}\right\rceil-3 \quad (8)$$
,

$$N = 2\cdot\left\lceil 0.5\cdot\left(\left(\frac{A_s}{15.5\cdot\beta\cdot(f_s-f_p)}\right)-\frac{\log_{10}(W))}{\beta\cdot(f_s-f_p)}+\log_{10}(W)-2\right)\right\rceil \quad (9)$$
,

$$M = \left\lceil\sqrt{\frac{A_s-4\cdot(\log_{10}(W)^2-8\cdot\log_{10}(W)-15)}{3.5}}+\frac{2.5\cdot(\log_{10}(W)}{3.5}-4\beta\cdot(f_s-f_p)\right\rceil+2\cdot \quad (10)$$

In applications of multirate cascaded structures with Farrow structure, Farrow structure serves to attenuate images of the regular FIR interpolator/decimator. So, the *Case B* specification should be used – (9) and (10).

6. DESIGN EXAMPLES

This part gives several examples that demonstrate how to utilize the formulae given, and concurrently we can estimate their performance. The following criteria are used to show this:

Case A *specifications*: The passband and stopband edges are at $f_p = 0.4F$ and at $f_s = 0.5F$; *Case* B *specifications*: The passband and stopband edges are at $f_p = 0.4F$ and at $f_s = 0.6F$; *Case* C *specifications*: The passband and stopband edges are at $f_p = 0.4F$ and at $f_s = 0.6F$.

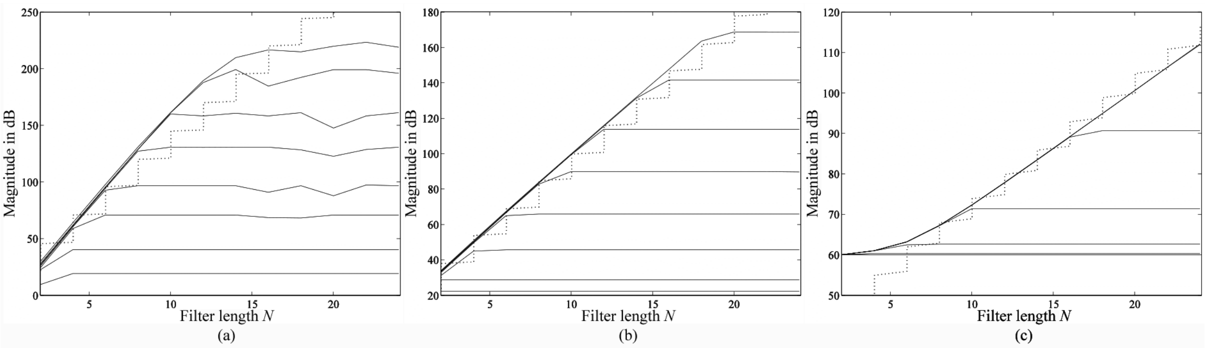

Fig. 3 – *Case B specifications*: The curves are shown for M equals 0 to 7. Dashed line is plot obtained from the estimation formula for N shown in [16]. The stopband edge is at $f_s = 1 – f_p$, passband edge and stopband weighting are at: (a) $f_p = 0.15F$ and $W = 0.1$; (b) $f_p = 0.3F$ and $W = 10$; (c) $f_p = 0.45F$ and $W = 1000$.

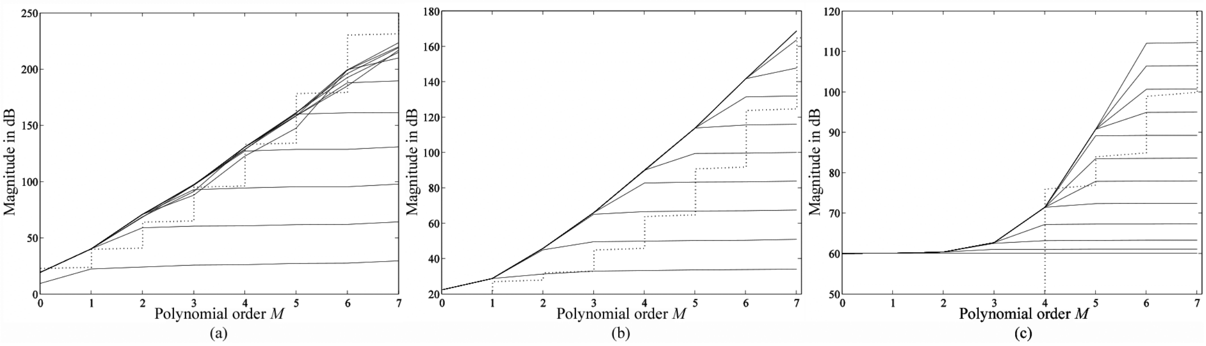

Fig. 4 – *Case B specifications*: The curves are shown for N equals 2 to 24. Dashed line is plot obtained from the estimation formula for M shown in [17]. The stopband edge is at f_s=1–f_p, passband edge and stopband weighting are at: (a) f_p=0.15F and W=0.1; (b) f_p=0.3F and W=10; (c) f_p=0.45F and W=1000.

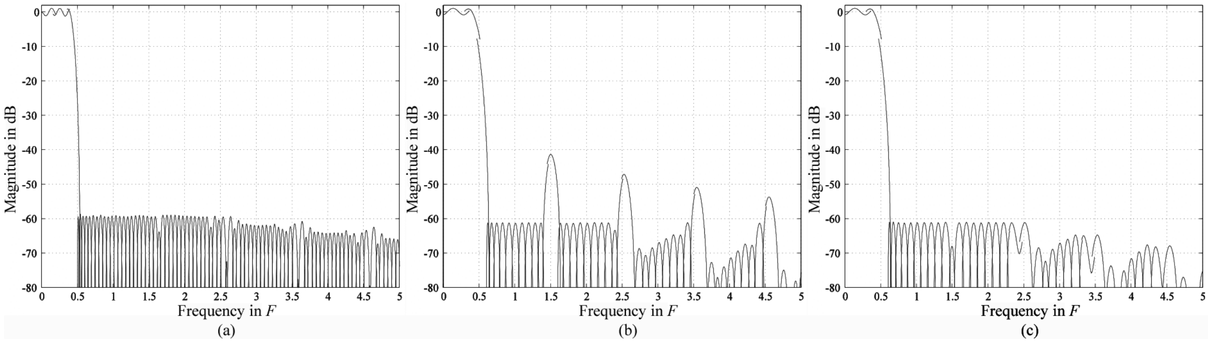

Fig. 5 – *Design Examples*: The frequency domain performance of filters whose parameters are estimated using the presented formulae. The filter specifications are stopband attenuation A_s=60 dB, passband weighting equal to unity and stopband weightings of W=100, passband edge f_p=0.4F. (a) *Case A* filter of length N=18, and M=4 with achieved A_s=59.5 dB and δ_p=0.1065; (b) *Case B* filter of length N=10, and M=4 with achieved A_s=61.3 dB and δ_p=0.0.0864; (c) *Case C* filter of length N=10, and M=4 with achieved A_s=61.2 dB and δ_p=0.0866.

In each case, filters should be designed in the minimax sense with stopband attenuation A_s = 60 dB, the passband weighting equal to unity and stopband weightings of W = 100.

Figure 5 shows the performance for *Case A*, *B* and *C* on sections (a), (b), and (c), respectively. By the presented formulae above, parameters N and M have been found: N = 18 and M = 4 for *Case A* and, and N = 10 and M = 4 for *Case B* and *C* as well. The estimation formulas appear to be rather good since they estimate the border performance for the provided set of criteria.

7. CONCLUSIONS

The estimate equations for the number N of polynomial segments and the polynomial order M in the minimax optimization are presented in this paper, which is a more general and precise result than in any previous research since the formulae include all three specifications (*Case A*, *B* and *C*) for the Farrow structure, the prolonged Farrow, and the Farrow structure in cascaded multirate filter as well. Filter designers can save time by using formulae, which provide beginning values for N and M and they can be used to estimate Farrow-based filter implementation costs for the given set of requirements, and also implementation costs of composed sampling rate converters containing Farrow, for example, in optimal factorization for multistage decimation (interpolation). Future work will include estimation formulas for least-mean-square optimization.

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