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MULTIDIMENSIONAL LOGARITHMIC SPHERICAL MU-LAW QUANTIZATION

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Keywords: Logarithmic spherical quantization; Quantization noise; Signal-to-noise ratio; µ-law.

Several researchers considered logarithmic spherical signal quantization. The design of a logarithmic spherical quantizer in *k* dimensions is considered in this paper. Logarithmic quantization is here used for vectors consisting of samples in sphere coordinates. The combination of logarithmic and spherical quantization (LSQ) is an efficient multi-dimensional quantization method at a high dynamic range, by preserving the original signal as close as possible. The signal-to-noise ratio (SNR) and its dependence on the sphere dimension at lower bitrates are derived and discussed in this paper.

1. INTRODUCTION

The area of digital signal processing (DSP) has experienced explosive development in the last four decades primarily due to the advancements and availability of digital signal processors (abbreviated as DSPs). Nowadays, DSP systems such as mobile smartphones and high-speed wireless networking systems become an integral part of daily life. Analog sensors measure analog signals in response to different physical processes that occur in an analog way, such as signal amplitude. Signal processing can be carried out in one of two ways: the analog or digital domain. To represent an analog signal in the digital domain, it is required that a digital signal is created by processes of sampling and quantizing (digitizing) the analog signal. There are several reasons why an analog signal should be converted into a digital form. The fundamental reason is that digital processing permits programmability, contrary to analog processing.

Quantization is a very important phase in the digitization process [1]. During the decades' quantization became an integral part of all DSP systems. While an analog signal is continuous, a digital signal is quantized in both time and amplitude. It is the basic step in the signal transformation from analog form to digital form. DSP includes the mathematical treatment of digital signals to obtain useful information.

Quantization determines how many different representation levels one digital signal has. Quantization can be scalar or vector, depending on if samples are quantized individually or in a block. Each sample of the signal in scalar quantization is quantized separately, while in vector quantization several samples are jointly quantized. Vector quantizers have better performances than scalar quantization in one way because they can usually achieve better SNR for the same bit rate [2]. On the other hand, the main obstacle of vector quantizers is high complexity, which increases with the increase of the dimension. The choice of the quantizer is usually based on the statistical characteristics of a signal which should be digitally processed.

The structure of this paper is as follows. Section II considers the basic principles of logarithmic spherical quantization. In Section III µ-law signal compression is described. Section IV concerning quantization noise and SNR concludes this paper.

2. LOGARITHMIC SPHERICAL QUANTIZATION

Logarithmic spherical quantization (LSQ) is a special kind of vector quantization based on representing a vector formed by samples in spherical coordinates and searching for a quantization cell on a *k*-dimensional sphere while quantizing the corresponding radius with respect to a logarithmically spaced codebook [3,4].

The radius is quantized separately from the surface of the sphere, whereas the SNR (signal-to-noise ratio) is independent of the radius within the logarithmic area. Spherical quantization is investigated in more detail in [5] and [6]. A reasonable balance between quantization performance and coding complexity is presented by Conway and Sloane [7].

LSQ is a quantizing method that combines, on the one hand, good properties of multidimensional logarithmic quantization at a moderate complexity, and on the other side can achieve coding gain. The LSQ method achieves a trade-off in the rate-distortion sense. Its characteristics are high dynamic range and low structural delay of a few sample periods, usually up to 16.

The evaluation of this method can be estimated by an SNR gain because it is the most used objective quality measure [8]. One important property of spherical quantization is that it offers some quantization gain, even if there don't exist statistical dependencies among samples combined with a vector. However, the main drawback of vector quantization is that it increases the complexity of the system, with a little or marginal gain.

In this paper, we propose µ-law ("mu-law") logarithmic quantization for the radius of the sphere r (magnitude), because it provides almost constant SNR in a wide range of input variance. The SNR is independent of the signal variance and probability density function (pdf). We decided to consider µ-law quantization because it is used for many years in telecommunications [9].

3. MU-LAW SIGNAL COMPRESSION

The problem with the logarithmic compression is that it is most sensitive to spectral parts with the worst power, *i.e.*, where the SNR is usually the lowest. Furthermore, values below 1 can cause problems with the float number range of the computer. A solution for this problem can be to use the logarithmic function $\ln(1+r)$, instead of $\ln r$, because 1 is a minimum threshold to which critical values should be set.

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The μ-law compression function $g(x)$ of an input signal x, on which the quantizer`s design is based, is defined with [10–12]:

$$g(x) = x_{\max} \cdot \text{sign}(x) \cdot \frac{\ln\left(1 + \frac{\mu |x|}{x_{\max}}\right)}{\ln(1+\mu)},$$

$$\text{where } \text{sign}(x) = \begin{cases} +1, \textit{for } x > 0 \\ 0, \textit{for } x = 0 \\ -1, \textit{for } x < 0 \end{cases} \quad (1)$$

The value x_{max} is the amplitude of the maximum load of the quantizer which defines the support range of the logarithmic quantizer, whereas μ is a non-dimensional compression factor. The support region means the interval where quantization errors are low or at least bounded. Accurate and fast estimation of the support region that minimizes distortion of the signal is very useful in quantizer design.

In our approach $x_{max} = 1$, because normalization is made before the companding energy. Furthermore, sign(x) = 1, because we operate with the positive values of input signal x. Therefore, the compression function becomes

$$g(x) = \frac{\ln(1+\mu(x))}{\ln(1+\mu)} = \frac{1}{\ln(1+\mu)} \cdot \ln(1+\mu(x)). \quad (2)$$

If we replace 1/[ln(1+ μ)] with a scale factor r_0, which depends only on values of μ, as well as μ(x) with r, we obtain a new compression function

$$g(x) = r_0 \cdot \ln(1+\mu(x)) = g(r) = r_0 \cdot \ln(1+r). \quad (3)$$

μ-law compression is flexible because it depends only on the parameter μ, and by choosing the appropriate value of μ the quantizer can be adjusted to different types of signals. In the logarithmic area, the derivative of the compression function applied for the radius r is:

$$g'(r) = \frac{dg(r)}{dr} = r_0 \cdot \frac{1}{1+r} = \frac{r_0}{1+r}. \quad (4)$$

If we assume M quantization intervals for the radius, the width of a quantization cell in radial direction Δ_r is as follows [12,13]:

$$\Delta_r = \frac{r_0}{M \cdot g'(r)} = \frac{r_0}{M \cdot \frac{r_0}{1+r}} = \frac{1+r}{M}. \quad (5)$$

Let Δ is the width of a quantization cell (field) that covers the k-dimensional sphere surface:

$$\Delta_r = \Delta \cdot r. \quad (6)$$

We choose the radius $r = 1$, for the sake of simplicity. Therefore, we consider that the quantization cells cover the whole surface of the unit sphere. This simplification is possible because the SNR does not depend on the radius due to logarithmic quantization. It follows from eqs. (5) and (6):

$$\Delta = \frac{2}{M} \rightarrow M = \frac{2}{\Delta}. \quad (7)$$

We can see that the number of the quantization intervals M (horizontal layers) available for the radius of the unit sphere is a function only on the width of quantization cells.

With S_k we denote the surface area, or the (k–1)-dimensional content of a sphere. It is uniformly covered with (k–1)-dimensional cubes and they contribute to a region of Δ^{k-1} to the whole surface area of the sphere. Therefore, the surface of the unit sphere, subdivided into C (number of the quantizing cells on the surface of the k-dimensional unit sphere) equal cubes, is as follows [7]

$$S_k = C \cdot \Delta^{k-1}. \quad (8)$$

The width of quantization cells Δ is approximately equal in all dimensions. The number of quantization levels N^k (sublayers) available per quantization step can be optimally split into C and M:

$$N^k = C \cdot M \rightarrow C = \frac{N^k}{M}. \quad (9)$$

By replacing M from (7) and C from (9) in eq. (8), the value S_k can be transformed as

$$S_k = \frac{N^k}{M} \cdot \Delta^{k-1} = \frac{N^k \cdot \Delta^{k-1}}{\frac{2}{\Delta}} = \frac{N^k \cdot \Delta^{k-1} \cdot \Delta}{2} = \frac{(N \cdot \Delta)^k}{2} \quad (10)$$

$$\rightarrow 2S_k = (N \cdot \Delta)^k. \quad (11)$$

From eq. (11) we obtain that the width of the quantization cell on the surface of the unit sphere is

$$\Delta = \frac{1}{N}(2S_k)^{1/k}. \quad (12)$$

The combination of eqs. (8) and (12) yields the number of the quantization cells on the surface as

$$C = \frac{S^k}{\left[\frac{1}{N}(2S_k)^{1/k}\right]^{k-1}} = \frac{S_k \cdot N^{k-1}}{(2S_k)^{\frac{k-1}{k}}}, \quad (13)$$

i.e.

$$C = S_k^{\frac{k+1-k}{k}} \cdot N^{k-1} \cdot 2^{\frac{1-k}{k}} = 2^{\frac{1-k}{k}} \cdot N^{k-1} \cdot S_k^{1/k}. \quad (14)$$

If $k \rightarrow \propto$, the number of cubes C becomes

$$C = \frac{N^{k-1}}{2}. \quad (15)$$

By replacing eq. (15) into (9), we obtain that the number of the quantization intervals M depends only on the total available number of bits per sample N as

$$M = \frac{N^k}{C} = \frac{N^k}{\frac{N^{k-1}}{2}} = 2N. \quad (16)$$

4. QUANTIZATION NOISE AND SNR

If we have enough quantization cells in k dimensions, the approximation of a uniformly distributed quantization error within those quantization cells can be utilized [14]. Having that in mind, we can use the usual term for the noise variance σ^2 within these cells [10]:

$$\sigma^2 = \frac{\Delta_r^{\,2}}{12} \cdot k . \tag{17}$$

The signal-to-noise ratio (SNR), defined as the squared amplitude of a signal r divided by the noise variance σ^2, is a widely applied measure for quantifying objective signal quality and it usually serves as a comparison between different algorithms:

$$\mathrm{SNR} = \frac{r^2}{\sigma^2} = \frac{r^2}{\frac{\Delta_r^{\,2}}{12} \cdot k} = \frac{r^2}{\frac{\Delta^2 \cdot r^2}{12} \cdot k} = \frac{12}{\Delta^2 \cdot k} . \tag{18}$$

By replacing Δ from (12) into (18) we obtain for SNR:

$$\mathrm{SNR} = \frac{12}{\left[\frac{1}{N}\left(2S_k\right)^{1/k}\right]^2 \cdot k} = \frac{12}{k} \cdot N^2 \cdot \frac{1}{\left(2S_k\right)^{2/k}} = \Omega(k) \cdot N^2 . \tag{19}$$

The SNR is dependent on N intervals per sample, and not on the variance of the signal in considered area. The term $\Omega(k)$ can be considered as the loss regarding the rate-distortion bound:

$$\Omega(k) = \frac{12}{k} \cdot \frac{1}{\left(2S_k\right)^{2/k}} . \tag{20}$$

It has been known, at least since the 19th century that the surface of the sphere is

$$S = S_k \cdot r^{k-1} . \tag{21}$$

If $r = 1$, the surface of the unit sphere S_k is equal to the surface of the sphere S [7,15,16]

$$S_k = k \cdot V_k , \tag{22}$$

where V_k is the volume of the unit sphere in k dimensions (k-dimensional content). It can be calculated according to [7,13] as

$$V_k = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)}, \text{ if } k \text{ is even,} \tag{23}$$

where Γ is the Gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, \mathrm{d}t, \textit{ while } \Gamma(x+1) = x! . \tag{24}$$

Therefore,

$$V_k = \frac{\pi^{k/2}}{(k/2)!} . \tag{25}$$

We will consider later what happens if k is odd. eq. (25) avoids the use of the Gamma function. We replace V_k with eq. (22) and the surface of the unit sphere can be described as

$$S_k = \frac{k \cdot \pi^{k/2}}{(k/2)!} . \tag{26}$$

The values of volume and surface area of an object are sometimes potentially confused, and instead, the term k-dimensional content of an object is often used. To approximate factorial from eq. (26) we will use the Stirling approximation:

$$x! \approx \sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x , \tag{27}$$

$$\to (k/2)! \approx \sqrt{2\pi \cdot \frac{k}{2}} \cdot \left(\frac{k}{2e}\right)^{k/2} = \sqrt{\pi k} \cdot \frac{k^{k/2}}{(2e)^{k/2}} = \frac{\sqrt{\pi} \cdot k^{\frac{k+1}{2}}}{(2e)^{k/2}} . \tag{28}$$

Inserting approximated factorial from (28) into (26), and then transforming S_k into (20) we obtain

$$\Omega(k) = \frac{12}{k} \cdot \frac{1}{\left[2 \cdot \frac{k \cdot \pi^{k/2}}{(k/2)!}\right]^{2/k}} = \frac{12}{k} \cdot \frac{1}{\left[\frac{2k \cdot \pi^{k/2}}{\frac{\sqrt{\pi} \cdot k^{\frac{k+1}{2}}}{(2e)^{k/2}}}\right]^{2/k}} , \tag{29}$$

i.e.,

$$\Omega(k) = \frac{12}{\pi} \cdot \frac{\left[\sqrt{\pi} \cdot k^{\frac{k+1}{2}}\right]^{2/k}}{\left[(2e)^{k/2}\right]^{2/k} \cdot k^{\frac{k+2}{k}} \cdot 2^{2/k}} = \frac{6}{\pi e} \cdot \frac{\pi^{1/k} \cdot k^{\frac{k+1}{k}}}{k^{\frac{k+2}{k}} \cdot 2^{2/k}} . \tag{30}$$

Finally, after some re-ordering $\Omega(k)$ has the following form:

$$\Omega(k) = \frac{6}{e} \cdot \frac{\pi^{\frac{1-k}{k}} \cdot k^{\frac{k+1-k-2}{k}}}{2^{2/k}} = \frac{6}{e} \cdot \frac{\pi^{\frac{1-k}{k}}}{k^{1/k} \cdot 2^{2/k}} . \tag{31}$$

When $k \to \infty$

$$\Omega(k) \to \frac{6 \cdot \pi^{-1}}{e} = \frac{6}{\pi e} ,$$

i.e., $10 \log \Omega(k) \to -1.55$ dB.

This is the loss of the rate-distortion bound obtained by the suboptimal quantization cells ($k \to \infty$), instead of quantization of the surface of a k-dimensional unit sphere. That means that by using logarithmic spherical quantization, we can compensate for the loss up to a margin of 1.55 dB. Further improvements of the system beyond this margin of 1.55 dB are not possible at all under the given requirements. It is well known that the rate-distortion function is an upper bound on the minimum rate at a given distortion.

On the other side, if k is odd, we should repeat the previous calculus, but with a different value for V_k [7,17]:

$$V_k = \frac{2^k \cdot \pi^{\frac{k-1}{2}} \cdot \left(\frac{k-1}{2}\right)!}{k!} . \tag{32}$$

Now we are applying the Stirling approximation for $k!$ from (27) and on the other side for $[(k\text{-}1)/2]$, as follows

$$\left(\frac{k-1}{2}\right)! \approx \sqrt{2\pi \cdot \frac{k-1}{2}} \cdot \left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}} = \sqrt{\pi(k-1)} \cdot \left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}} . \tag{33}$$

Now, S_k from eq. (22) can be calculated with the value for V_k from eq. (32) and approximation from eq. (33) as

$$S_k = k \cdot \frac{2^k \cdot \pi^{\frac{k-1}{2}} \cdot \sqrt{\pi(k-1)} \cdot \left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}}}{\sqrt{2\pi k} \cdot \frac{k^k}{e^k}} = \frac{k^{1/2} \cdot 2^k \cdot \pi^{\frac{k-1}{2}} \cdot (k-1)^{k/2} \cdot e^{\frac{2k-k+1}{2}}}{2^{k/2} \cdot k^k} = 2^{k/2} \cdot \pi^{\frac{k-1}{2}} \cdot e^{\frac{k+1}{2}} \cdot k^{\frac{1-2k}{2}} \cdot (k-1)^{k/2}. \quad (34)$$

By inserting (34) into (20) we obtain

$$\Omega(k) = \frac{12}{k} \cdot \frac{1}{\left[2 \cdot 2^{k/2} \cdot \pi^{\frac{k-1}{2}} \cdot e^{\frac{k+1}{2}} \cdot k^{\frac{1-2k}{2}} \cdot (k-1)^{k/2}\right]^{2/k}} = \frac{12}{k \cdot \left[2^{\frac{k+2}{k}} \cdot \pi^{\frac{k-1}{k}} \cdot e^{\frac{k+1}{k}} \cdot k^{\frac{1-2k}{k}} \cdot (k-1)\right]} = \frac{12}{(k-1) \cdot k^{\frac{1-k}{k}} \cdot 2^{\frac{k+2}{k}} \cdot \pi^{\frac{k-1}{k}} \cdot e^{\frac{k+1}{k}}}. \quad (35)$$

If $k \to \infty$ then $\Omega(k) \to \frac{6}{\pi e}$.

Interestingly, we can notice that there is the same loss bound when k is even and odd: $10\log \Omega(k) \to -1.55$dB

When $k \to \infty$, the logarithm of eq. (19) yields

$$10\log \mathrm{SNR} = 10\log \Omega(k) + 10\log N^2 = 20\log N - 1.55\,\mathrm{dB} \quad (36)$$

Let R [bpp] denotes the average bitrate and $N = 2^{Rk}$ – is the total available number of bits per sample (code vectors) [16]. To be implemented in a communication system, N code vectors must be identified by strings of length $R{\cdot}k$ which are transmitted through a communication channel. Then, SNR can be calculated as

$$10\log SNR = 10\log \Omega(k) + 20\log 2^{R\cdot k} = 10\log \Omega(k) + 20 \cdot Rk \cdot (\log 2) = 10\log \Omega(k) + 6.02 \cdot Rk. \quad (37)$$

However, $\Omega(k)$ has different values when k is even or odd. We compared in Fig. 1 SNRs for different lower bit rates R (0.2, 0.4, and 0.6 bpp) and even sphere dimensions k.

5. CONCLUSION

This paper presented one method for SNR calculation, which combines gains from logarithmic and spherical quantization. The logarithmic spherical quantization is applicable to any number of dimensions k. The derived performance expressions may be used to decide if obtained results are of value in a particular application.

The results presented here for the logarithmic spherical quantizers show that they can be efficient in several known applications, such as speech coding and other source coding problems, numerical evaluations of integrals on spheres, or the minimum energy configuration of a sphere for astronomy and physics applications.

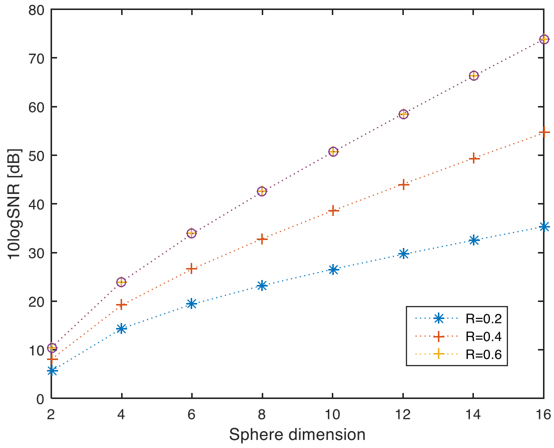

Fig. 1 – Signal-to-noise ratio (SNR) vs. sphere dimension.

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