CONSTRUCTAL LAW CONFERENCE – CLC2024
Design in Nature and Evolution
Romanian Academy, Bucharest, 10–11 October 2024

Proceedings of CLC2024
pp. 133–136

CONSTRUCTAL LAW APPLIED TO PROOF MAXIMUM WORK PRINCIPLE. CONSEQUENCES ON CONVEXITY AND NORMAL RULE OF BULKING PLASTICITY & SURFACE FRICTION POTENTIAL TOGETHER WITH ESTIMATION OF REAL THERMO-MECHANICAL POWER DURING CONTINUUM MEDIA FLOWS

ADINEL GAVRUS[a]

National Institute of Applied Sciences – INSA Rennes, 20 Av. des Buttes de Coesmes, 35700 Rennes, France
adinel.gavrus@insa-rennes.fr

Following the author's previous studies, this scientific article uses the proof of the “Principle” of Maximum Work (PMW), used in metals plasticity theory and surface tribology, through a more general mathematical framework starting from the fundamental Constructal Law. In this paper, the author focuses on the mathematical proof of convexity and normal rule properties of both plastic and friction flow criteria.

Keywords: Maximum Work “Principle” proof; Bulk plasticity; Surface friction; Potential convexity and normal rule; Thermo-mechanical power estimation.

1. INTRODUCTION

Starting from previous author studies, this research uses the proof of the “Principle” of Maximum Work (PMW) defined in metals plasticity theory and surface tribology [1] through a more general mathematical framework starting from the fundamental Constructal Law [2,3], and the Virtual Powers Principle (VPP) defined because of the momentum stresses equilibrium. The Constructal Law postulates the natural tendency of any finite-size system to evolve towards an optimal space-time configuration, minimizing losses and entropy generation.

Regarding a material deformation during a forming or flow process, under specified boundary/loadings conditions, the real mechanical variables defining the flow (velocities, stresses, strain, and strain rate) are those that minimize the sum of dissipated bulk deformation and contact surfaces friction powers written in terms of all other virtual and admissible states. It is then show that PMW becomes a Theorem [4-8] and can be applied to all continuous media (solid, fluid, mushy state, plasma) and any type of materials.

An equivalent form has also been defined for contact friction (isotropic and anisotropic). This paper focuses on the mathematical proof of convexity and normal rule laws of both plastic and friction potential.

DOI: https://doi.org/10.59277/CLC/2024.47

2. GENERAL THEORETICAL FRAMEWORK

Starting from the Constructal Law which postulate the tendency of all system to search a such configuration evolution or flow minimizing losses, as it has already been applied by authors in previous research works [4-8], in the case of continuum media flow, this one is obtained by minimizing the functional defining the virtual dissipated or lost power P^*_d, *i.e.*:

$$P_d = \mathrm{Min}\left(P^*_d\right) \text{ with } P^*_d = \iiint_{\Omega} \left[\sigma^*\right] : \left[\dot{\varepsilon}^*\right] dV + \iint_{\partial\Omega'} -\vec{\tau}^* \cdot \Delta\vec{v}^* \, dS' + \iint_{\partial\Omega^*} -\vec{T}^d \cdot \vec{v}^* \, dS" . \quad (1)$$

Here, ":" denotes the matrix contracted product equivalent to the vector's scalar product, where * represents all admissible virtual values corresponding to all kinematic and mechanical variables: $\vec{v}^*$ – virtual velocities vector, $\left[\dot{\varepsilon}^*\right]$ – virtual strain rate tensor (symmetric part of the velocity vector gradient matrix), $\left[\sigma^*\right]$ - virtual Cauchy stress tensor, $\vec{\tau}^*$ – virtual shear stress vector occuring on surfaces interfaces and $\vec{T}^d$ – imposed stress-force vector regarded also as constraint loading.

Then the real solution of the flow in terms of real stresses and flow velocities is obtained by minimizing P^*_d corresponding to the optimization problem (1).

Using the well known Virtual Power Principle, it has proven the First PMW Theorem:

$$\left(\left[\sigma^*\right]-\left[\sigma\right]\right) : \left[\dot{\varepsilon}^*\right] \geq 0, \forall\left[\sigma^*\right], \Phi_p\left(\left[\sigma^*\right]\right)-\sigma_0 = 0 \, ; -\left(\vec{\tau}^* - \vec{\tau}\right)\cdot\Delta\vec{v}^* \geq 0, \forall\vec{\tau}^*, \Psi_f\left(\vec{\tau}^*\right)-\tau_f = 0 , \quad (2)$$

In a reverse form it can be written the First PMW Theorem using the equivalent form:

$$\left(\left[\sigma\right]-\left[\sigma^*\right]\right) : \left[\dot{\varepsilon}\ \right] \geq 0, \forall\left[\sigma^*\right], \Phi_p\left(\left[\sigma^*\right]\right)-\sigma_0 = 0 \, ; -\left(\vec{\tau} - \vec{\tau}^*\right)\cdot\Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f\left(\vec{\tau}^*\right)-\tau_f = 0 , \quad (3)$$

Here $\Phi_p\left(\left[\sigma^*\right]\right)$ it is the plastic criterion (as isotropic Von-Mises or anisotropic Hill one) and $\Psi_f\left(\vec{\tau}^*\right)$ represents the friction potential (circular for isotropic friction and elliptic for anisotropic friction where friction coefficient is function of sliding directions).

2.1. Proof of Plastic and Friction Potential Normal Rule

In conformity with the relationship (2) concerning the stress state, for any infinitesimal stress variation $\pm d\left[\sigma\right]=\left[\sigma^*\right]-\left[\sigma\right]$ with $\Phi_p\left(\left[\sigma \pm d\sigma\right]\right)=\Phi_p\left(\left[\sigma\right]\right)=\sigma_0$ it is obtained:

$$\mp\left[d\left[\sigma\right]/\left\|d\left[\sigma\right]\right\|\right]:\left[\dot{\varepsilon}\right] \geq 0 \Rightarrow \left[d\left[\sigma\right]/\left\|d\left[\sigma\right]\right\|\right]:\left[\dot{\varepsilon}\right]=0 \quad (4)$$

A first Taylor development gives $\Phi_p\left(\left[\sigma \pm d\sigma\right]\right)=\Phi_p\left(\left[\sigma\right]\right) \pm d\left[\sigma\right]:\partial\Phi_p/\partial\left[\sigma\right]$, then:

$$\pm\left[d\left[\sigma\right]/\left\|d\left[\sigma\right]\right\|\right]:\partial\Phi_p/\partial\left[\sigma\right]=0 \quad (5)$$

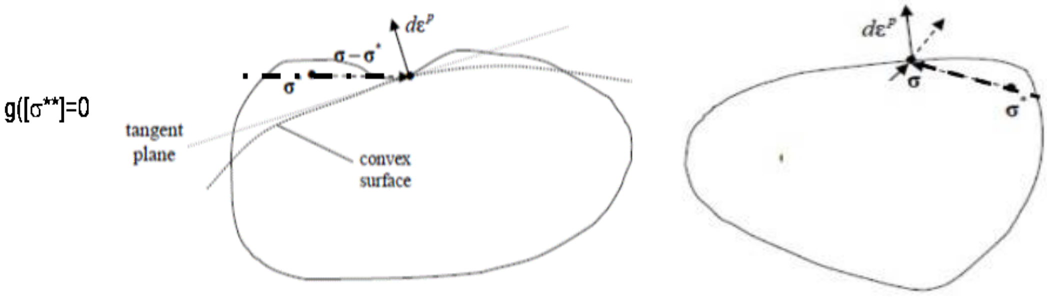
Using equations (4) and (5), it can be conclude on the proportionality of $\partial\Phi_p / \partial[\sigma]$ with $[\dot{\varepsilon}]$ and the named normal rule law of plastic potential it is obtained, *i.e.* :

$$d\varepsilon^p / dt = [\dot{\varepsilon}] = \lambda_p \partial\Phi_p / \partial[\sigma], \lambda_p \geq 0 \tag{6}$$

In a similar way, concerning the virtual friction shear stress can also be prove the friction normal rule law, *i.e.:*

$$\Delta\vec{v} = -\lambda_f \partial\Psi_f / \partial\vec{\tau}, \lambda_f \geq 0 \tag{7}$$

Figure 1 shows the potential shape, the plastic strain hyper-vector, and the stress states.



(a) (b)

Fig. 1 – Schema of real or virtual stress states and the plastic potential surface shape: (a) different positions of infinitesimal plastic deformation vector $d\varepsilon^p = [\dot{\varepsilon}]dt$; (b) plastic potential surface and tangent plane.

2.2. **Proof of Plastic and Friction Potential Convexity**

All tangent hyper-planes on the plastic potential curves can be defined by $g([\sigma^{**}]) = \Phi_p([\sigma]) - \Phi_p([\sigma^{**}]) + ([\sigma^{**}] - [\sigma]) : \partial\Phi_p / \partial[\sigma] = 0$, when applying inequality given by equation (3) written in the reversible form of the first PMW Theorem (2), for any real stress states situated on a potential curve, together with the corresponding normal rule law i.e. $([\sigma] - [\sigma^*]) : [\dot{\varepsilon}] \geq 0 \Rightarrow ([\sigma] - [\sigma^*]) : \partial\Phi_p / \partial[\sigma] \geq 0$, it is obtained $g([\sigma^*]) \leq 0$.

Consequently, the potential curve is located at any point inside the tangent plane's family. In conformity with one of the convexity definition form, the convex shape of stress criteria is then proven. Similar conclusion can be obtain concerning the convexity of friction potential.

Finally, using this convexity property, the two inequalities of equation (2) can be extended to all virtual stress or friction shear respecting $\Phi_p([\sigma^*]) - \sigma_0 \leq 0$ $\Psi_f(\vec{\tau}^*) - \tau_f \leq 0$ corresponding to the points inside the closed potential curve. It is then possible to define the final form of the PMW Theorem for both stress and friction state:

$$([\sigma] - [\sigma^*]) : [\dot{\varepsilon}] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) - \sigma_0 \leq 0 \tag{8}$$

$$-(\vec{\tau} - \vec{\tau}^*) \cdot \Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) - \tau_f \leq 0 \tag{9}$$

3. **RESULTS**

Previous author's works concerning computation applications for plane compression [4], cylindrical crushing [5], direct extrusion [8], and anisotropic contact surface friction [6] show the feasibility of the above PMW theorem and its consequences.

Accurate analytical predictions are obtain to estimate average forming process power P, with an error ε less than 5%, by formula:

$$P = \left(\tilde{P} + \tilde{\tilde{P}}\right)/2 \quad (10)$$

where $\varepsilon = 100 * (\tilde{P} - \tilde{\tilde{P}})/2P$.

Here $\tilde{P}$ represents the Upper Bound Power estimation and $\tilde{\tilde{P}}$ the Lower Bound Power estimation, obtained from applying the PMW theorem to specific predefined subspaces of virtual stresses, kinematic velocities and boundaries-loadings conditions [4-8].

4. **DISCUSSION AND CONCLUSIONS**

This paper proves plastic and friction criteria convexity with customary rule laws corresponding to plastic flow of any continuum media. The performed theory has been valid by previous works of the author through comparisons of numerical results obtained from Finite Element Modelling (FEM) with analytical computations based on the PMW theorem and its consequences.

REFERENCES

1. Avitzur B., *Metal Forming: Processes and Analysis*, New-York, McGraw-Hill, 1968
2. Bejan A., *Shape and Structure: From Engineering to Nature*, UK, Cambridge Univ., 2000.
3. Bejan A., S. Lorente, *Design with Constructal Theory*, New Jersey, John Willey & Sons, 2008.
4. Gavrus A., The Maximum Work Principle Regarded as a Consequence of an Optimization Problem Based on Mechanical Virtual Power Principle and Application of Constructal Theory, *AIP Conf. Proc*., Vol. 1896, pp. 100009-1-6, 2017.
5. Gavrus A., Application of Constructal Theory to Write Mechanical Maximum Work Principle and Equilibrium State of Continuum Media Flow as a Solution of a Variational Optimization Problem, *Int. Revue IJMO*, **8**, *4*, pp. 227–231 (2018).
6. Pîrva E., Tudor A., Gavrus A., Stoica N., Cananau S., Some aspects regarding the influence of the anisotropy of an AA2021-T351 rolled thick plate on its tribological behaviour, *Mechanics* & *Industry*, **20**, *6* (2019).
7. Gavrus A., Computational Framework Concerning the Formulation of Maximum Work Principle used in Plasticity, Materials Forming and Tribology as a Consequence of a Variational Optimization Problem Defined From the Constructal Law, *IOP Conf. Series: Mat. Sci. and Eng*., Vol. 968, 2020.
8. Gavrus A., Use of the Constructal Law to Prove the Maximum Work "Principle" Defining Plasticity, Contact Friction and Continuum Media Flow Using a Variational Optimization Framework. Application to Extrusion Power Estimation, *Int. CLC 2023 Conf.*, Turin, Italy, 21–22 September 2023.